ON CIRCULAR MOTIONS OF THE FROUDE PENDULUM

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We shall investigate a mechanical system whose vibrations are controlled by the differential equation

$$\ddot{x} + \alpha \dot{x} + f(x) = N \operatorname{sign}(\Omega - \dot{x}) \tag{1}$$

where α, N and Ω are positive constants, and $f(x)$ is an odd periodic function with a continuous derivative, which in the interval $[-\pi, \pi]$ has the following two roots:

$$xf(x) > 0 \quad \text{near } x = 0, \qquad f(\pi) = f(0) = 0 \tag{2}$$

When $f(x) = \sin x$ then the differential equation (1) controls vibrations of the Froude pendulum [1].

We shall investigate a solution of (1) in the form $\dot{x} = \dot{x}(x)$, periodic with respect to x. This solution represents circular motions of the Froude pendulum with completely defined $F(\Omega - \dot{x}) = N \operatorname{sign}(\Omega - \dot{x})$. This last function characterizes friction of the shaft in the bearing of the pendulum, where Ω is the angular speed of the shaft. Investigating only the torsional motion as obtained from (1), we shall assume that

$$N > \max f(x) \tag{3}$$

Equation (1) is equivalent to the system of differential equations

$$\dot{x} = y, \qquad \dot{y} = -\alpha y - [f(x) + N \operatorname{sign}(\dot{x} - \Omega)] \tag{4}$$

which can be written in the form of two systems, setting

$$N \operatorname{sign}(y - \Omega) = +N \ \text{ when } y \geqslant \Omega, \qquad N \operatorname{sign}(y - \Omega) = -N \ \text{ when } y < \Omega \tag{5}$$

Thus instead of investigating trajectories described by the system (4), we shall investigate trajectories described by the systems

$$\dot{x} = y, \qquad \dot{y} = -\alpha y - [f(x) + N] \tag{6}$$
$$\dot{x} = y, \qquad \dot{y} = -\alpha y - [f(x) - N] \tag{7}$$

The cylindrical phase space of the systems (6) and (7), equivalent to the investigated system (4), will be developed on the xy-plane. Trajectories will repeat themselves periodically on strips, 2π wide, along the y-axis. For this reason it is sufficient to investigate only one trajectory of the system (4) which is traced, for example, in the strip $-\pi \leqslant x \leqslant \pi$, $-\infty < y < \infty$, and the results of this investigation will apply to all the periodically repeated trajectories traced in the xy-plane.

The systems (6) and (7) under the conditions (2) and (3) have no singular points and no limit cycles of the first kind, which correspond to trajectories of these systems periodic with respect to t. All trajectories of both (6) and (7) when $t \to \infty$ approach asymptotically a unique stable limit cycle [1] of the second kind, which embraces the cylindrical surface of the phase space. This limit cycle for the system (6) is in the half-plane $y < 0$ and for the system (7) in the semi-plane $y > 0$. Consequently, the investigated system (4) has no singular points and no limit cycles of the first kind. All its trajectories are attracted by the stable limit cycle Y when $t \to \infty$. It should be mentioned that although the parameter Ω does not appear explicitly on the right-hand sides of Equations (6) and (7), nevertheless it exerts a considerable influence on the trajectories of the system (4). For this reason we must study variations of the trajectories of the system (4) when Ω varies.

In order to be specific, we shall consider only values $\Omega > 0$. It is obvious that the presence of the limit cycle of the second kind y_0 of the system (7) influences trajectories of the system (4), and among others, it influences the ratio of Ω to the extrema of $y_0(x)$, that is, to the solution of the system (7), periodic with respect to x. Consequently, we will investigate in detail the character of trajectories of the system (7).

We introduce the curve of monotonicity

$$y = \frac{N - f(x)}{\alpha} \equiv \Phi(x) \tag{8}$$

This curve, together with the x-axis, divides the xy-plane into regions where the derivative dy/dx of the system (7) has the same sign. The periodic solution $y_0(x)$ of (7) has its extrema on the curve of monotonicity.

The boundaries for the variations of Ω are the extrema of the curves

$y = y_0(x)$ and $y = \phi(x)$. Let the values of Ω satisfy the inequality

$$\Omega \geqslant \max \Phi(x) \tag{9}$$

Then it can be easily shown that the critical line $y = \Omega$ is the line of no contact [1] for the system (4). This means that the limit cycle Y of the investigated system (4) and the limit cycle y_0 of the system (7) equal each other. All the trajectories of (4) approach y_0 asymptotically when $t \to \infty$, and in the half-plane $y < \Omega$ they merge with the trajectories of the system (7). On the line $y = \Omega$ the trajectories have a "kink", and when $y > \Omega$ they continue as the trajectories of the system (6). Similar cycles of the second kind will be called "ordinary", because the trajectories approach them asymptotically as $t \to \infty$. Let us have further

$$\max y_0(x) < \Omega < \max \Phi(x) \tag{10}$$

The line $y = \Omega$ intersects the curve of monotonicity (8). When at some instant of t the point $[x(t), y(t)]$ of the system (4) coincides with any segment of the line $y = \Omega$ such that $y < \phi(x)$, then it cannot leave the segment and has to move along it as t increases. We shall call such a segment "segment of capture" of the system (4). For the values of Ω under consideration we have only one segment of capture of the system (4) in the strip $-\pi \leqslant x \leqslant \pi$.

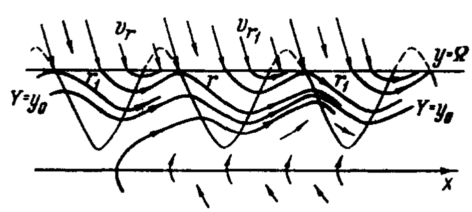

Fig. 1.

Let us denote by r this trajectory $x = x(t)$, $y = (t)$ of the system (4) which exists from the right end of the segment of capture (Fig. 1) in the strip $-\pi < x < \pi$. This means that for the trajectory r there exists $t = T < \infty$, such that the point $[x(T), y(T)]$ belonging to the trajectory of (4) coincides with the right boundary point of the segment of capture in the strip $-\pi < x < \pi$. Let us study the trajectory when $t > T$. A curve merging with the trajectory r will be denoted by $y = r(x)$. A trajectory which exits from the right end of the segment of capture (Fig. 1) in the strip $\pi < x < 3\pi$ (which is a periodic repetition of the trajectory r) will be denoted by r_1, and so on.

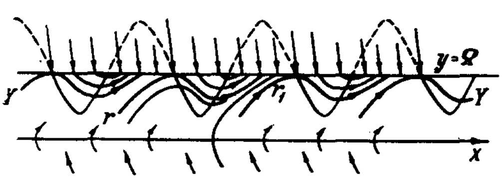

Fig. 2.

Every segment of capture determines the existence of a whole family of trajectories of (4) which coincide for some finite values of t with the trajectories r, r_1, and so on. Such families of trajectories will be denoted by v_r, v_{r_1}, and so on.

From the inequality (10) it follows that $\Omega > \max y_0(x)$, which means that the limit cycle y_0 of the system (7) is also the limit cycle of the system (4). The trajectory r of (4) which is at the same time a trajectory of (7) is above its limit cycle and has the property that $r(x) > r(x + 2\pi)$ for all x in the region where the curve $y = r(x)$ exists. It means that $r \in v_{r_1}$. Consequently, the trajectory r approaches asymptotically the cycle $Y \equiv y_0$ of the system when $t \to \infty$.

Thus, when Ω satisfies the inequality (10), the ordinary cycle Y of (4) equals the cycle y_0 of (7), just as in the case when Ω satisfied the inequality (9). The case (10) differs from the case (9) in that the existence of segments of capture in (4) has resulted in the families of the trajectories v_r, v_{r_1}, and so on, which merge with the trajectories r, r_1 and so on, respectively, and approach asymptotically the cycle when $t \to \infty$ (Fig. 1).

We shall assume now that Ω satisfies the equality

$$\Omega = \max y_0(x) \tag{11}$$

In this case, as in previous cases, the cycle Y of the system (4) equals the cycle y_0 of the system (7), but it is not an ordinary cycle now, in the sense of our previous definition. Indeed, for $y < y_0$, as in the previous case, all the trajectories of (4) being also trajectories of (7) approach asymptotically the cycle as $t \to \infty$, but for $y > y_0$ each trajectory of the system (4) merges with the limit cycle at finite instants of t. We shall call a cycle of (4) of the second type a "special cycle" if there is a class of trajectories of the system (7) merging with that cycle at finite value of time t. Suppose that Ω satisfy the inequality

$$\min \Phi(x) < \Omega < \max y_0(x) \tag{12}$$

This means that in the strip $-\pi \leqslant x \leqslant \pi$ the system (4) has only one segment of capture. We shall denote by r_0 the trajectory of the system (7) which merges with the trajectory r when $y < \Omega$. Since $\Omega < \max y_0(x)$, the trajectory r_0 belonging to the system (7) and located below the cycle y_0 will approach asymptotically the cycle y_0 when $t \to \infty$. Consequently, for the trajectory r_0 the inequality $r_0(x) < r_0(x + 2\pi)$ will be satisfied for every x in this region of the half-plane $y > 0$ where the curve $y = r_0(x)$ exists. For the above reason the trajectory r_0 must intersect the segment of capture (Fig. 2) in the strip $\pi < x < 3\pi$, meaning that $r \in v_{r_1}$.

Thus, when Ω satisfies the inequality (12), then in the strip $-\pi \leqslant x \leqslant \pi$ the limit cycle of the system (4) will consist of the trajectory r and of a certain part of the segment of capture. This pattern is repeated periodically on all the strips of width 2π in the xy-plane. The cycle Y will then be a special cycle and all the trajectories of the system (4) will merge with this cycle. Finally, let us assume that Ω satisfies the inequality

$$0 < \Omega \leqslant \min \Phi(x) \tag{13}$$

It can be easily shown that in this case the critical line $y = \Omega$ is also the line of capture and also a special cycle of the system (4). We have proved thus the following theorem:

Theorem 1. All the trajectories of the system (4) satisfying the conditions (2) and (3) when $t \to \infty$, attract each other through the limit cycle of a second kind, either asymptotically or merging with it at finite instants of time.

It is obvious that the value $\Omega = \max y_0(x)$, where y_0 is a periodic solution of the system (7), is a bifurcation value of the parameter Ω for the system (4). When Ω satisfies the inequality $0 < \Omega < \max y_0(x)$, then the system (4) has an ordinary limit cycle.

Theorem 2. If the system (4) satisfying the conditions (2) and (3) satisfies also the inequality $0 < \Omega < N/a$, then the system has a special cycle; when the system satisfies the inequality

$$\Omega > \frac{N + \max f(x)}{\alpha}$$

then it has an ordinary cycle.

The proof of this theorem follows from the estimate

$$\frac{N}{\alpha} < \max y_0(x) < \max \frac{N - f(x)}{\alpha}$$

Here N/a is the mean value of the function $y_0(x)$ which is the periodic solution of the system (7) in the interval $[-\pi, \pi]$.

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